Partial Fraction Decomposition

Working with rational expressions

Goal. Rewrite one rational expression as a sum of simpler fractions. First, make sure the fraction is proper, then factor the denominator, choose the correct setup, and solve for the unknown constants.

Before You Begin

- The degree of the numerator must be less than the degree of the denominator. If it is not, use polynomial division first.
- Factor the denominator completely over the real numbers.
- Use a different capital letter for each unknown numerator.

Choose the Correct Setup

Denominator factor	Example factor	Required fraction term
Single linear	$x-4$	$\frac{A}{x-4}$
Repeated linear factors	$(x-4)^3$	$\frac{A}{x-4}+\frac{B}{(x-4)^2}+\frac{C}{(x-4)^3}$
Non-factorable Quadratic Denominator	x^2+9	$\frac{Ax+B}{x^2+9}$
Repeated Non-factorable Quadratic	$(x^2+9)^2$	$\frac{Ax+B}{x^2+9}+\frac{Cx+D}{(x^2+9)^2}$

The Four Steps

1. Factor: Factor the denominator completely.

2. Set up: Write one fraction for every factor and every repeated power.

3. Clear denominators: Multiply every term by the least common denominator.

4. Solve and check: Substitute convenient x-values or match coefficients, then recombine the fractions to check.

Worked Example

Decompose $\frac{5x+1}{x^2+x-2}$

Step 1 Factor

Factor the denominator into two distinct linear factors: $x-1$ and $x+2$.

Step 2: Set Up

$$\frac{5x+1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

Step 3: Clear the Denominators

Multiply every term by $(x-1)(x+2)$:

$$5x+1 = A(x+2) + B(x-1)$$

Step 4: Solve

Substitution Table

Choose x	Substitute	Result
$x=1$	$5(1)+1 = A(1+2)+B(1-1)$ $5+1 = A(3)+B(0)$ $\frac{6}{3} = \frac{3A}{3}$	$A=2$
$x=-2$	$5(-2)+1 = A(-2+2)$ $+B(-2-1)$ $-10+1 = A(0)+B(-3)$ $\frac{-9}{-3} = \frac{-3B}{-3}$	$B=3$

Final Answer

$$\frac{5x+1}{(x-1)(x+2)} = \frac{2}{x-1} + \frac{3}{x+2}$$

Quick check: Combining the two fractions gives $\frac{2(x+2)+3(x-1)}{(x-1)(x+2)} = \frac{5x+1}{(x-1)(x+2)}$.

Case IV: Repeated Non-Factorable Quadratic: Additional worked example

A trinomial example

Example

$$\frac{x^3 + 3x^2 + 9x + 2}{(x^2 + 2x + 5)^2}$$

1. Check the quadratic

For $x^2 + 2x + 5$, calculate the discriminant: $b^2 - 4ac = 2^2 - 4(1)(5) = -16 < 0$.

Thus, the quadratic is not factorable over the real numbers. Also, $x^2 + 2x + 5 = (x + 1)^2 + 4 > 0$, so the original domain is R.

2. Write the complete decomposition form

Include each power of the quadratic, with a separate linear numerator:

$$\frac{x^3 + 3x^2 + 9x + 2}{(x^2 + 2x + 5)^2} = \frac{Ax + B}{x^2 + 2x + 5} + \frac{Cx + D}{(x^2 + 2x + 5)^2}$$

3. Clear denominators and match coefficients

$$x^3 + 3x^2 + 9x + 2 = (Ax + B)(x^2 + 2x + 5) + (Cx + D).$$

Multiply $(Ax + B)(x^2 + 2x + 5)\ and\ add\ (Cx + D) = Ax^3 + 2Ax^2 + 5Ax + Bx^2 + 2Bx + 5B + Cx + D$

Combine like terms

$$x^3 + 3x^2 + 9x + 2 = Ax^3 + (2A + B)x^2 + (5A + 2B + C)x + (5B + D)$$

Matching the coefficients: Compare the polynomials on both sides of the equation and set the coefficients of corresponding powers of the variable equal (as $Ax^3 = 1x^3\ and\ 3x^2 = (2A + B)x^2$.

Matching Coefficients Table

Left Side	Right Side	Matching & Calculation
$1x^3$	Ax^3	$A = 1$
$3x^2$	$(2A + B)x^2$	2(1) +B=3, B=3-2, B= 1
$9x$	$(5A + 2B + C)x$	$(5(1) + 2(1) + C) = 9$
2	$(5B + D)$	$5 + 2 + C = 9, C = 9 - 7, C = 2$
		$(5(1) + \mathrm{D}) = 2, D = 2 - 5, D$ =-3

4. **Write the answer and verify**

$$\frac{x^3+3x^2+9x+2}{(x^2+2x+5)^2}=\frac{x+1}{x^2+2x+5}+\frac{2x-3}{(x^2+2x+5)^2}$$

Practice

For each problem, write the correct setup and solve for the constants. Show your work in the space provided.

1. $\frac{3x+5}{x(x+1)}$
2. $\frac{7x-1}{(x-2)(x+3)}$
3. $\frac{2x^2+5x+1}{x(x+1)^2}$
4. $\frac{x^2+3x+5}{(x-1)(x^2+4)}$
5. $\frac{x^3+2x^2+7x+1}{(x^2+1)^2}$

Answer Key

No.	Setup	Decomposition
1	$3x+5=A(x+1)+Bx$	$\frac{5}{x}-\frac{2}{x+1}$
2	$7x-1=A(x+3)+B(x-2)$	$\frac{13}{5(x-2)}+\frac{22}{5(x+3)}$
3	$\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$	$\frac{1}{x}+\frac{1}{x+1}+\frac{2}{(x+1)^2}$
4	$\frac{A}{x-1}+\frac{Bx+C}{x^2+4}$	$\frac{9}{5(x-1)}+\frac{-\frac{4}{5}x+\frac{11}{5}}{x^2+4}$
5	$\frac{x^3+2x^2+7x+1}{(x^2+1)^2}=\frac{Ax+\mathrm{B}}{x^2+1}+\frac{Cx+D}{(x^2+1)^2}$	$\frac{x^3+2x^2+7x+1}{(x^2+1)^2}=\frac{x+2}{x^2+1}+\frac{6x-1}{(x^2+1)^2}$

AI prompts:

5. Provide a similar problem and show a step-by-step solution with no answer displayed.
6. In the correct setup section, why is setup B different from setup D?